Alternative random models of the zeros of the Riemann zeta function

Alex Dobner

October 13, 2025

Background: the Riemann zeta function

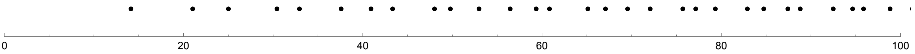
Riemann zeta function: $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ for $\text{Re}(s) > 1$.
Analytic continuation: $\zeta(s)$ extends to a meromorphic function on $\mathbb{C}$ with a simple pole at $s = 1$.
Nontrivial zeros: $\zeta(s)$ has infinitely many zeros in the critical strip $0 < \text{Re}(s) < 1$.
Riemann Hypothesis (RH): All nontrivial zeros of $\zeta(s)$ lie on the critical line $\text{Re}(s) = \frac{1}{2}$.

Imaginary parts of the first few nontrivial zeros:

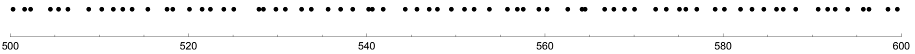
$$14.1347\ldots, 21.0220\ldots, 25.0108\ldots, 30.4249\ldots, 32.9351\ldots, \ldots$$

Imaginary parts of the zeros

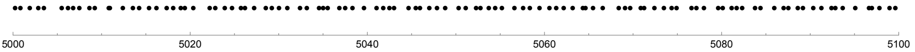
Write the zeros as $\frac{1}{2} + i\gamma$. (RH $\iff$ $\gamma \in \mathbb{R}$.)
Locations of all $0 < \gamma < 100$:



$500 < \gamma < 600$:



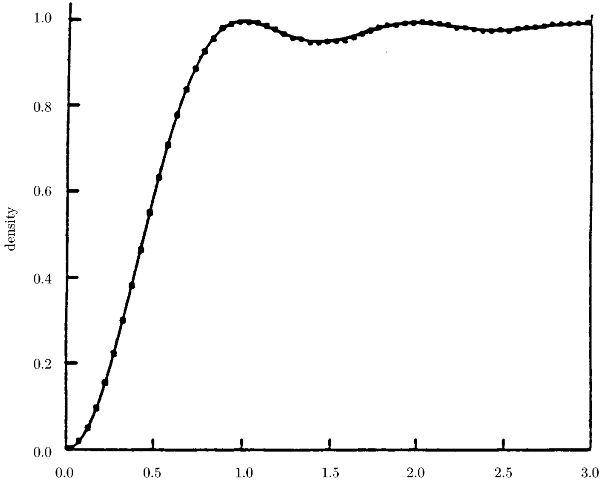
$5000 < \gamma < 5100$:



When $\gamma \approx T$, the mean spacing between consecutive zeros is $\sim \frac{2\pi}{\log(T/2\pi)}$.

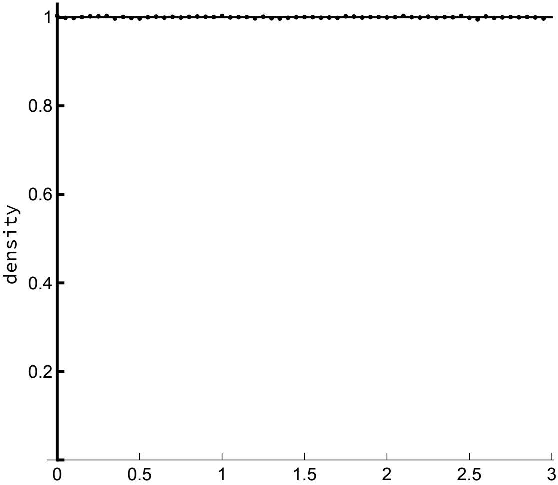
How are the γ's distributed? Are they independent of one another? **No!**

Normalized gap distribution of zeta zeros

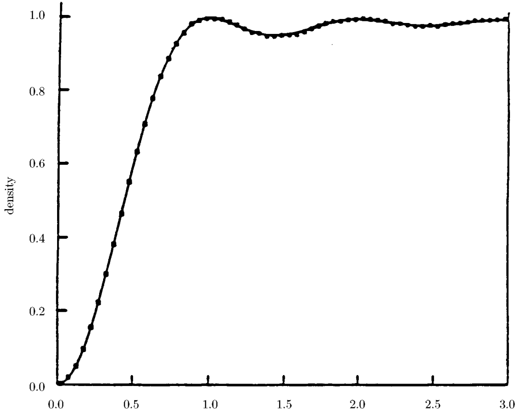


Distribution of normalized distances between pairs of 8×10^6 zeros near height $T = 10^{20}$. (Image by Katz & Sarnak using data of Odlyzko.)

Compare this with:



Distribution of distances between pairs of 10^7 reals chosen independently and uniformly in $[0, 10^7]$.



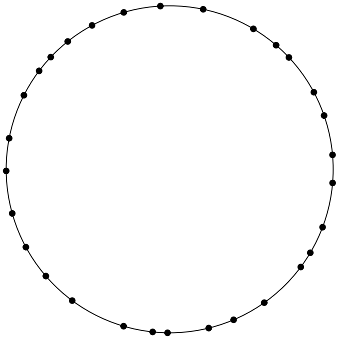
Distribution of normalized distances between pairs of 8×10^6 zeros near height $T = 10^{20}$. (Image by Katz & Sarnak using data of Odlyzko.)

Montgomery's pair correlation conjecture (1973): As $T \to \infty$ the empirical distribution depicted above converges to $1 - \left(\frac{\sin(\pi x)}{\pi x}\right)^2$.

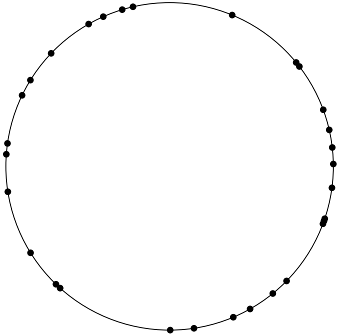
Background: random unitary matrices

Circular Unitary Ensemble (CUE): The group $U(N)$ of $N \times N$ unitary matrices, with Haar probability measure.

What do the eigenvalues of a typical $U \sim \mathrm{CUE}(N)$ look like?



Eigenvalues of a random unitary matrix ($N = 30$).



30 points chosen independently and uniformly on the unit circle.

Normalized gap distribution of CUE eigenangles

Eigenangles: Given $U \in U(N)$, write eigenvalues as $e(\theta_1), \ldots, e(\theta_N)$ where $\theta_j \in [0,1)$ and $e(x) := e^{2\pi i x}$.
What is the expected distribution of the pairwise differences $\theta_j - \theta_k$?

For fixed $0 \leq \alpha \leq \beta \leq N$, consider the random variable

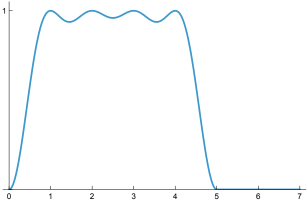
$$\mathcal{N}_{\alpha,\beta} = \#\{(j,k) : 1 \leq j \neq k \leq N \text{ and } \frac{\alpha}{N} \leq (\theta_j - \theta_k \bmod 1) < \frac{\beta}{N}\}.$$

Theorem (Dyson)

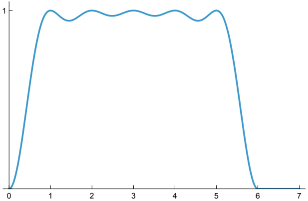
$$\mathbb{E}_{CUE(N)} \frac{1}{N} \mathcal{N}_{\alpha,\beta} = \int_\alpha^\beta \left(1 - \left(\frac{\sin(\pi x)}{N \sin(\pi x / N)}\right)^2\right) dx$$

As $N \to \infty$, the integrand tends to $1 - \left(\frac{\sin(\pi x)}{\pi x}\right)^2$.
This is the same density that appears in Montgomery's conjecture.

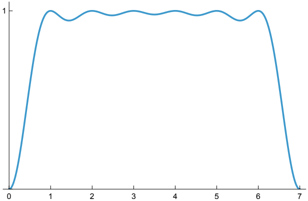
N = 5



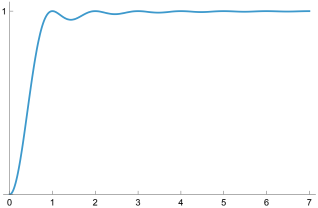
N = 6



N = 7



$$\xrightarrow[N\to\infty]{}$$

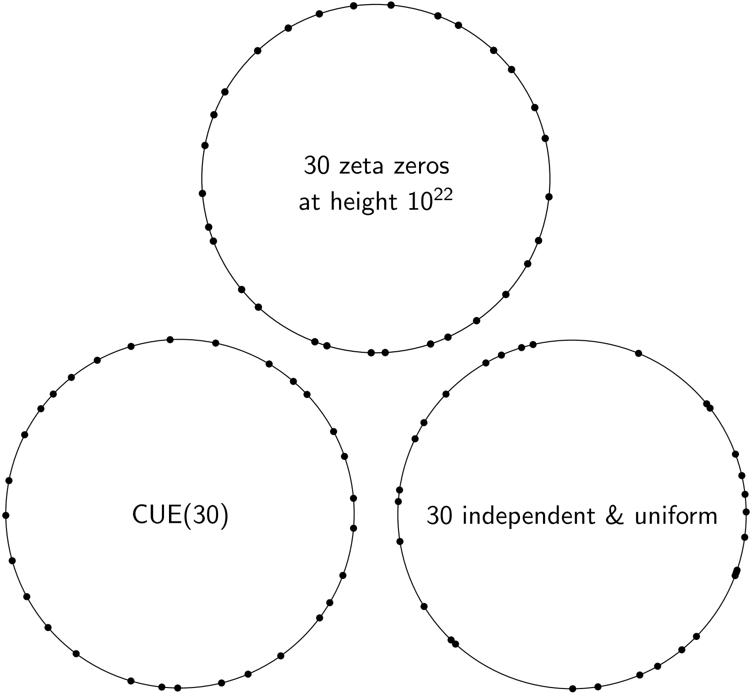


CUE hypothesis (informal)

Let T be large, and let $1 \leq N \ll \log T$. Choose $t \in [T, 2T]$ uniformly at random. Take the first N zeta zeros above height t and wrap them around the unit circle. This N-tuple of points behaves statistically like the eigenvalues of a random CUE(N) matrix.

This type of model was studied from a statistics perspective by **Diaconis and Coram (2003)**.

Keating and Snaith (2000): used a CUE model to conjecture asymptotics for $\int_T^{2T} |\zeta(\frac{1}{2} + it)|^{2k}\, dt$.


30 zeta zeros
at height 10^{22}
CUE(30)
30 independent & uniform

Higher correlations and the CUE hypothesis

What is known rigorously about correlations of zeta zeros?

- Montgomery (1973): pair correlation of zeta zeros agrees with CUE prediction for test functions whose Fourier transforms are supported in $(-1, 1)$.
- Rudnick and Sarnak (1996): n-point correlations of zeta zeros agree with CUE prediction for test functions whose Fourier transforms are supported in the set $\{(x_1, \ldots, x_n) : |x_1| + \cdots + |x_n| < 2\}$.

CUE hypothesis

The n-point correlations of the zeta zeros in $[T, 2T]$ agree asymptotically with the n-point correlations of CUE(N) matrix eigenvalues. (In the respective $T \to \infty$ and $N \to \infty$ limits.)

Alternative hypotheses for zeta zeros

CUE hypothesis

The n-point correlations of the zeta zeros in $[T, 2T]$ agree asymptotically with the n-point correlations of CUE(N) matrix eigenvalues. (In the respective $T \to \infty$ and $N \to \infty$ limits.)

We can consider the CUE hypothesis to be the **null hypothesis** for the distribution of zeta zeros.

What kinds of **alternative hypotheses** are there that are consistent with the our rigorous knowledge about zeta zeros (i.e. Rudnick-Sarnak)? Are there models which have...

- no small gaps between consecutive zeros? Yes $\begin{cases} \text{Tao} \\ \text{Lagarias \& Rodgers} \end{cases}$
- no large gaps between consecutive zeros?
- a positive probability of multiple zeros?

Arithmetic significance of small gaps

The question of small gaps is significant because of a connection to Siegel zeros.

Folklore (cf. Watkins 2019): If $L(s, \chi)$ has a "bad" Siegel zero, then the zeros of $\zeta(s)L(s, \chi)$ up to a certain height are all on the critical line and are approximately in an arithmetic progression.

Conrey & Iwaniec (2002): If sufficiently many gaps between consecutive zeta zeros are less than half the average gap, then there are no Siegel zeros.

Finite point process models

Fix a positive integer N.

Let μ be a probability measure on the N-torus: $[0,1)^N$. Suppose μ is invariant under permuting components of the N-tuple.

Equivalent ways of thinking about μ:

- μ describes a **point process** on $[0,1) \cong S^1$. That is, it gives a way of picking a random multiset of N points in $[0,1)$.
- μ describes a conjugation invariant probability measure on $U(N)$. (Conjugacy classes of $U(N)$ correspond to multisets of N points in S^1. Choose the eigenangles according to μ, then conjugate by $V \sim \mathrm{CUE}(N)$.)

Example: CUE(N) measure.
Notation: $U \sim \mathrm{CUE}(N)$ and $(\theta_1, \ldots, \theta_N) \sim \mathrm{CUE}(N)$.

Example: CUE measure

Theorem (Weyl)

Suppose $(\theta_1, \ldots, \theta_N) \sim CUE(N)$*. The joint distribution of* $(\theta_1, \ldots, \theta_N)$ *is given by*

$$\frac{1}{N!} \prod_{1 \le j < k \le N} \left| e(\theta_j) - e(\theta_k) \right|^2 d\theta_1 \cdots d\theta_N.$$

Analogue of Rudnick-Sarnak correlations

Theorem (Diaconis & Shahshahani 2003)

Let $U \sim CUE(N)$. Let $a_1, \ldots, a_k$ and $b_1, \ldots, b_k$ be nonnegative integers such that $\sum_{j=1}^k j\, a_j \leq N$. Then

$$\mathbb{E}_{CUE(N)} \prod_j (\mathrm{tr}\, U^j)^{a_j} (\overline{\mathrm{tr}\, U^j})^{b_j} = \prod_j j^{a_j} a_j!$$

if $a_j = b_j$ for all j, and zero otherwise.

The restriction $\sum_{j=1}^k j a_j \leq N$ corresponds in a precise way to the Fourier support restriction for test functions in the Rudnick-Sarnak result on n-point correlations of zeta zeros.

Hence, the “alternative models” for the zeta zeros are probability measures that satisfy the Diaconis-Shahshahani result above.

Definition

Given a probability measure μ on $U(N)$ that is conjugation invariant, we'll say that μ *mimics* CUE(N) if it satisfies the Diaconis-Shahshahani moment conditions (i.e. if one can replace CUE(N) by μ in the statement of their theorem).

Note: the mimicry condition is equivalent to saying that the Fourier coefficients of the eigenangle measure μ must agree with those of CUE(N) eigenangle measure in a certain range.

Does there exists a μ that mimics CUE(N) and has...

1. no “small gaps” between consecutive eigenangles almost surely?
2. no “large gaps” between consecutive eigenangles almost surely?
3. a positive probability of eigenvalues that have multiplicity ≥ 1?

Tao proved that the answer to (1) is yes. I don't know the answer to (2) or (3).

ACUE measure

Definition

Draw $(\theta_1, \ldots, \theta_N)$ from the discrete set $\{0, \frac{1}{2N}, \ldots, \frac{2N-1}{2N}\}^N$ according to the probability density function

$$\frac{1}{(2N)^N}\frac{1}{N!}\prod_{1\leq j<k\leq N}|e(\theta_j) - e(\theta_k)|^2.$$

This probability measure on $[0,1)^N$ is the ACUE(N) eigenangle measure.

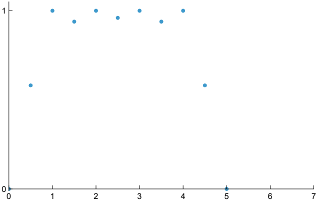
Note: Tao also adds a random universal rotation the angles θ_j. For our results this won't make much difference.

Theorem (Tao 2019)

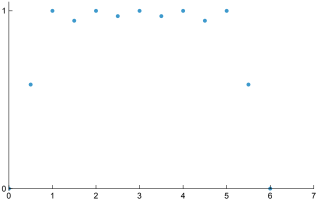
ACUE(N) mimics CUE(N).

All gaps between ACUE eigenangles are a positive half-integer multiple of the mean spacing $\frac{1}{N}$.

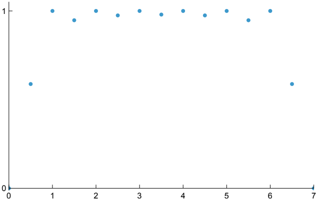
N = 5



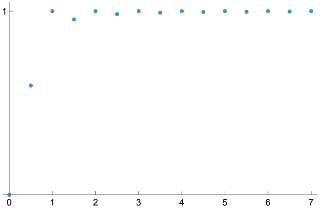
N = 6



N = 7



$$\xrightarrow[N\to\infty]{}$$



Results on ACUE(N): particle-hole duality

Unlike for CUE(N), the distribution of eigenvalues of ACUE(N) has a certain duality property.

Every ACUE(N) eigenvalue configuration consists of N points out of all the $(2N)$th roots of unity. Hence, there are also N "holes" (the $(2N)$th roots of unity that are not eigenvalues).

Theorem (D.)

The distribution of the holes of ACUE(N) is the same as the distribution of the eigenvalues.

Arithmetic significance: the ACUE model for zeta zeros is consistent with the presence of Siegel zeros, which forces the zeros of $\zeta(s)L(s,\chi)$ to lie approximately in an arithmetic progression. The hole distribution corresponds to the distribution of zeros of $L(s,\chi)$.

Particle-hole duality

Proof of theorem:

Given a configuration $\theta_1, \ldots, \theta_N$ of ACUE(N) eigenangles, let $\psi_1, \ldots, \psi_N$ denote the corresponding "hole" eigenangles. It suffices to show that the ACUE(N) density at $(\theta_1, \ldots, \theta_N)$ is the same as the ACUE(N) density at $(\psi_1, \ldots, \psi_N)$. This follows from two facts:

Fact 1: The ACUE(N) density of a configuration depends only on the pairwise differences $\theta_j - \theta_k$ (mod 1).
Fact 2: The pairwise differences $\theta_j - \theta_k$ (mod 1) and $\psi_j - \psi_k$ (mod 1) are the same multisets.

Fact 1 is clear from the definition of ACUE(N). Fact 2 follows from a simple Fourier analytic argument.

Facts about CUE

Theorem (Rains 1997, 2000): Suppose $U \sim \mathrm{CUE}(N)$. Then the following hold:

High powers of U	Let k be an integer such that $\lvert k \rvert \geq N$. Then the eigenvalues of U^k are distributed as N independent and uniform points on the circle.
Low powers of U	Let k be a divisor of N. Then the eigenvalues of U^k are distributed like the union of the eigenvalues of k independent matrices selected from $\mathrm{CUE}(N/k)$.
High moments of tr U	Let k be a positive integer. Then $$\mathbb{E}_{\mathrm{CUE}(N)} \lvert \operatorname{tr} U \rvert^{2k} = \lvert S_k(12 \cdots (N+1)) \rvert$$ where $S_k(12 \cdots (N+1))$ denotes the set of permutations of $\{1, \ldots, k\}$ that do not contain an increasing subsequence of length $N+1$.

Facts about ACUE

Theorem (D.): Suppose $U \sim \text{ACUE}(N)$. Then the following hold:

High powers of U	For any integer k, let k_0 be the unique integer in $(-N, N]$ such that $k_0 \equiv k \pmod{2N}$. The eigenvalues of U^k have the same distribution as $U^{\lvert k_0 \rvert}$.
Low powers of U	Let k be a divisor of N. Then the eigenvalues of U^k are distributed like the union of the eigenvalues of k independent matrices selected from $\text{ACUE}(N/k)$.
High moments of tr U	Let k be a positive integer. Then $\mathbb{E}_{\text{ACUE}(N)} \lvert \operatorname{tr} U \rvert^{2k} = \lvert S_k(N \cdots 1\,(N+1),\ 12 \cdots (N+1)) \rvert$ where $S_k(N \cdots 1\,(N+1), 12 \cdots (N+1))$ denotes the set of permutations of $\{1, \dots, k\}$ that do not contain any subsequence whose relative order is the same as $N \cdots 1\,(N+1)$ or $12 \cdots (N+1)$.

Proofs of ACUE results

High powers of U: Because the eigenvalues of U are $(2N)$th roots of unity, we have $U^{2N} = I$. This implies U^k is periodic with period $2N$. There is a further symmetry coming from the fact that $U^k \sim U^{-k}$.

Low powers of U: The proof is Fourier-analytic and is similar to Rains's proof of the CUE case. Suppose μ is a probability measure on $(\frac{1}{2N}\mathbb{Z}/\mathbb{Z})^N$. Given $(\theta_1, \ldots, \theta_N) \sim \mu$, we are interested in understanding the distribution of $(k\theta_1, \ldots, k\theta_N) \pmod 1$. There is a nice expression for the Fourier coefficients of this distribution in terms of the Fourier coefficients of μ.

The rest of the analysis involves applying the following expression for the ACUE(N) density:

$$\frac{1}{N!} \prod_{1 \le j < k \le N} |e(\theta_j) - e(\theta_k)|^2 = \sum_{\sigma \in S_N} \mathrm{sgn}(\sigma) \prod_{j=1}^{N} e((j - \sigma(j))\theta_j).$$

Proofs of ACUE results (cont.)

High moments of tr U: This proof is more involved...
Rains's proof for the CUE case uses representation theory and combinatorics. His result follows from two key equalities:

1. $$\mathbb{E}_{\mathrm{CUE}(N)} |\operatorname{tr} U|^{2k} = \sum_{\lambda \vdash k, \ell(\lambda) \leq N} (f^\lambda)^2$$
   where f^λ is the number of standard Young tableaux of shape λ.

2. $$\sum_{\lambda \vdash k, \ell(\lambda) \leq N} (f^\lambda)^2 = |S_k(12 \cdots (N+1))|$$
   where $S_k(12 \cdots (N+1))$ is the set of permutations of $\{1, \ldots, k\}$ that do not contain an increasing subsequence of length $N+1$.

The first equality comes from the representation theory of S_k and of $U(N)$. The second equality uses the Robinson-Schensted correspondence.

Proofs of ACUE results (cont.)

High moments of $\operatorname{tr} U$**:** For ACUE both equalities have analogues, but the proofs are different (both are combinatorial).

1. $$\mathbb{E}_{\mathrm{CUE}(N)} \left| \operatorname{tr} U \right|^{2k} = \sum_{\lambda \vdash k, \ell(\lambda) \leq N} (f_{\mathrm{cyl}}^{\lambda})^2$$
   where $f_{\mathrm{cyl}}^{\lambda}$ is the number of standard *cylindric* tableaux of shape λ (more specifically, these are cylindric tableaux with period (N, N)).

The proof uses the ACUE(N) density directly, along with a combinatorial argument involving nonintersecting lattice paths on a cylinder.

Proofs of ACUE results (cont.)

2. $$\sum_{\lambda\vdash k,\ell(\lambda)\leq N} (f_{\text{cyl}}^{\lambda})^2 = |S_k(N\cdots 1(N+1), 12\cdots(N+1))|$$

   where $S_k(N\cdots 1(N+1), 12\cdots(N+1))$ is the set of permutations of $\{1,\ldots,k\}$ that do not contain any subsequence whose relative order is the same as $N\cdots 1(N+1)$ or $12\cdots(N+1)$.

This proof uses a new variant of the Robinson-Schensted correspondence that is adapted for cylindric tableaux.

Other results

- The same methods can be used to get an expression for $\mathbb{E}_{\mathrm{ACUE}(N)} |\det(I-U)|^{2k}$ except now the combinatorial objects that appear are *semistandard* cylindric tableau.
  - For this result it is necessary to use Tao's definition of ACUE involving an additional random rotation.
- Some partial results on alternative models with no large gaps or with multiple eigenvalues

Connections to differential equations

- The CUE distribution has another interpretation in terms of (circular) Dyson Brownian motion which is defined in terms of a certain stochastic differential equation. Presumably there is a ACUE analogue using random walks rather than Brownian motion?
- Removing the stochastic part of Dyson Brownian motion gives a certain ODE which describes the evolution of zeros of polynomials under heat flow. Is there an ACUE analogue?

Dictionary

Zeta function at random height	Random CUE matrix
$t \sim \mathrm{Unif}([T, 2T])$	$U \sim \mathrm{CUE}(N)$
$x \mapsto \zeta(\frac{1}{2} + it - ix)$	$x \mapsto \det(I - Ue(x))$

Normalize so that average spacing of zeros is 1:

$x \mapsto \zeta(\frac{1}{2} + it - i\frac{2\pi x}{\log T})$	$x \mapsto \det(I - Ue(\frac{x}{N}))$

Logarithmic derivatives:

$x \mapsto \dfrac{2\pi i}{\log T} \displaystyle\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{1/2+it}} e(\frac{\log n}{\log T} x)$	$x \mapsto -\dfrac{2\pi i}{N} \displaystyle\sum_{j=1}^{\infty} \mathrm{tr}(U^j) e(\frac{j}{N} x)$

Collecting terms and equating: $\mathrm{tr}(U^j) \approx -\dfrac{N}{\log T} \displaystyle\sum_{n \in I_j} \frac{\Lambda(n)}{n^{1/2+it}}$

where $I_j = [T^{\frac{(j-1/2)}{N}}, T^{\frac{(j+1/2)}{N}})$.