KEEPING TRACK OF LEFT AND RIGHT ACTIONS

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1. Introduction

1.1. **Definitions.** The goal of this note is to record a useful mnemonic for remembering how group actions on sets lift to group actions on functions. This is described in Section 2. First we establish the definitions.

Let G be a group. A *left action* of G on a set X is a map $\cdot\colon G \times X \to X$ satisfying the conditions

(1) $e \cdot x = x$ for all $x \in X$
(2) $g \cdot (h \cdot x) = (gh) \cdot x$ for all $g, h \in G, x \in X$.

Similarly, we also have the notion of a *right action* of G on X. This is a map $\cdot\colon X \times G \to X$ satisfying the conditions

(1) $x \cdot e = x$ for all $x \in X$
(2) $(x \cdot g) \cdot h = x \cdot (gh)$ for all $g, h \in G, x \in X$.

We call X a left G-set or a right G-set depending on the type of action.

1.2. **Categorical perspective on group actions.** Note that the definition of a left group action looks a lot like the definition of function composition if we imagine g and h are functions). To clarify what's going on, it's helpful to state things in the language of category theory.

Any group G can be thought of a category $*_G$ with a single object and with morphisms that are exactly the elements of G. The morphism composition operation is defined to equal the group operation.[1] From this perspective a left group action is exactly the same thing as a functor $F\colon *_G \to \mathbf{Set}$. The G-set corresponding to F is just the single object in **Set** that's in the image of F. The fact that F is a functor means that it turns morphism composition in $*_G$ (i.e. the group operation) into morphism composition in **Set** (i.e. function composition).[2]

The reason that a functor $F\colon *_G \to \mathbf{Set}$ corresponds to a *left* action (as opposed to a right action) is because of how we define composition of functions in standard mathematical notation (or equivalently, because of how we define composition of morphisms in **Set**). Indeed, given two functions g, h their composition $g \circ h$ is given by the rule $(g \circ h)(x) = g(h(x))$. This is exactly the same as rule (2) for left actions.

At first glance, it is slightly strange that $g \circ h$ is defined to mean "apply h first, then g", but the reason for this is precisely because we write functions to the left of their arguments. Indeed, if we were to write the expression $g\ h\ x$ without any parentheses, then there is still no ambiguity about the meaning of the expression: apply h to x, then g). On the other hand, if we lived in a world where mathematicians wrote function application on the right-hand side (i.e. $x\ g\ h$) then naturally we'd define $g \circ h$ to mean "apply g first, then h". In this alternate world, a functor $F\colon *_G \to \mathbf{Set}$ would then correspond to a right action.

To go from the usual definition of function composition to the alternative definition, what we are essentially doing is passing from **Set** to the opposite category $\mathbf{Set}^{\mathrm{op}}$. Hence, we can say that a right action is just a functor $F : *_G \to \mathbf{Set}^{\mathrm{op}}$. Alternatively, a right action is a *contravariant* functor from $*_G$ to Set. That is, F satisfies $F(gh) = F(h) \circ F(g)$ for all morphisms in $*_G$.

[1] Note that the fact that G has inverses is not really relevant here. A category with a single object is equivalent to notion of a monoid. Amusingly, the suffix '-oid' is used in category theory when one wants to allow for multiple objects, so one can define a category to be a monoidoid.

[2] A *group representation* is a variant of this idea where the functor goes to a category of vector spaces. Once again, we could generalize this idea to monoid representations. Note that associativity is completely essential for making sense of these notions.

1.3. **Going between left and right actions.** For any group G, there is an isomorphism of categories between $*_G$ and the opposite category $*_G^{op}$. This is given by the mapping $g \mapsto g^{-1}$. Note that this is not the same as the trivial fact that *any* category is *anti*-isomorphic to its opposite category (via the "contravariant identity" functor).

This isomorphism allows one to convert any covariant functor $F\colon *_G \to \mathbf{Set}$ to a contravariant functor from $*_G$ to **Set** and vice versa. In other words: there is a natural bijective correspondence between left G actions and right G actions.

Concretely, given a left action $(g, x) \mapsto g \cdot x$, we define the right action by letting $x \cdot g := g^{-1} \cdot x$. Note that this correspondence relies on the fact that groups have inverses. We do not have such a bijective correspondence for general noncommutative monoids even though the definition of left and right actions make sense for these objects.

2. Lifting actions on sets to actions on functions

Suppose X and Y are sets. If either one of them is is a left or right G-set, then there is a natural way to make $\mathrm{Hom}(X, Y)$ into a G-set. See the following table for how this goes. (We'll use g to denote a group element and $\Psi\colon X \to Y$ to denote a function.)

Type of action:	Lifts to ... on $\mathrm{Hom}(X, Y)$	Via the definition...
left action on Y	a left action	$(g \cdot \Psi)(x) := g \cdot (\Psi(x))$
left action on X	a right action	$(\Psi \cdot g)(x) := \Psi(g \cdot x)$
right action on Y	a right action	$(\Psi \cdot g)(x) := \Psi(x) \cdot g$
right action on X	a left action	$(g \cdot \Psi)(x) := \Psi(x \cdot g)$

The first line corresponds to composing the covariant functor $*_G \to \mathbf{Set}$ (i.e. the left action on Y) with the covariant functor $\mathrm{Hom}(X, -)\colon \mathbf{Set} \to \mathbf{Set}$. The second line corresponds to composing the covariant functor $*_G \to \mathbf{Set}$ (the left action on X) with the contravariant functor $\mathrm{Hom}(-, Y)\colon \mathbf{Set} \to \mathbf{Set}$. Etc.

It is annoying to remember the rows of this table, but it turns out that there is a nice mnemonic. Looking at the first two rows in the table, note that the definition in the last column is simply a rearrangement of parentheses. Checking that these rows actually give valid left/right actions also amounts to rearranging parentheses. For example, for the second row we have

$$((\Psi \cdot g) \cdot h)(x) = (\Psi \cdot g)(h \cdot x) = \Psi(g \cdot (h \cdot x)) = \Psi((gh) \cdot x) = (\Psi \cdot (gh))(x).$$

Hence, to remember the second row of the table, we simply look at the expression $\Psi\ g\ x$ and we equate both ways of parenthesizing it. Similarly, to remember the first row, we do the same things with $g\ \Psi\ x$.

What about the other two rows of the table? For the third row we look at $x\ \Psi\ g$, and we take parentheses in two ways again and equate them. This time we interpret $(x\ \Psi)$ to mean "apply Ψ to x". Under this interpretation, verifying that we have a valid right action is once again a matter of moving parentheses around. Lastly, for the fourth row we do the same thing for $x\ g\ \Psi$.

Hence, we see that each of the four rows in the table comes from a different permutation of the three symbols x, g, and Ψ. What about the last two permutations? These are $g\ x\ \Psi$ and $\Psi\ x\ g$. Note that g is not adjacent to Ψ in either of these, so there is no way that equating the two ways of parenthesizing will define an action on $\mathrm{Hom}(X, Y)$. The resulting equations are still meaningful though. The statement $(g \cdot x)\ \Psi = g \cdot (x\ \Psi)$ is equivalent to saying that Ψ is an *equivariant map* from the left G-set X to the left G-set Y. Similarly, $(\Psi\ x) \cdot g = \Psi\ (x \cdot g)$ says that Ψ is an equivariant map between right G-sets.

2.1. **Actions involving g^{-1}.** Note that the expression g^{-1} does not appear anywhere in the table above. Consequently, everything above generalizes nicely to monoid actions as well. For completeness, we will now state some additional liftings that work when G is a group (in which case we can apply the correspondence described in Section 1.3). The resulting table is as follows.

Type of action:	Lifts to ... on $\mathrm{Hom}(X,Y)$	Via the definition...
left action on Y	a right action	$(\Psi \cdot g)(x) := g^{-1} \cdot (\Psi(x))$
left action on X	a left action	$(g \cdot \Psi)(x) := \Psi(g^{-1} \cdot x)$
right action on Y	a left action	$(g \cdot \Psi)(x) := \Psi(x) \cdot g^{-1}$
right action on X	a right action	$(\Psi \cdot g)(x) := \Psi(x \cdot g^{-1})$

Example: Let G be a compact group, and let $L^2(G)$ be the Hilbert space of square integrable functions (with respect to Haar measure on G). The left action of G on itself lifts to a left action of G on $L^2(G)$ using the second row of the second table. This is called the *left regular representation*. The right action of G on itself lifts to a left action of G on $L^2(G)$ using the fourth row of the first table. This is called the *right regular representation*.[3]

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[3]Note that as actions on $L^2(G)$, the left and right regular representations are both left actions. This is standard in representation theory (i.e. a group representation is a left action).