POISSON SUMMATION VIS-À-VIS EULER-MACLAURIN SUMMATION

ALEX DOBNER

1. Introduction

Suppose $f : \mathbb{R} \to \mathbb{C}$ is some relatively nice function and we want to compute or estimate the sum

$$\sum_{n\in\mathbb{Z}} f(n).$$

Two methods for this that get used frequently in analytic number theory are

(1) the Poisson summation formula, and
(2) the Euler-Maclaurin formula.

In this note, I'd like to record how to relate these two.

1.1. Notation.

Let $e(x) \coloneqq e^{2\pi i x}$. Then the Fourier transform of f is

$$\hat{f}(u) \coloneqq \int_{\mathbb{R}} f(x)e(-xu)\,dx.$$

For a periodic function $F \colon \mathbb{R} \to \mathbb{C}$ with period 1, the Fourier transform is a function $\hat{F} : \mathbb{Z} \to \mathbb{C}$ where

$$\hat{F}(n) = \int_{[0,1)} f(x)e(-nx)\,dx.$$

2. The summation formulas

First let's state each formula and sketch the proofs. Both are usually stated in a more general way, but the generalized versions make it more difficult to see the connection between the two. To avoid any technicalities in the proofs, let's assume that f is a Schwartz function.

Theorem 1 (Poisson summation formula)**.**

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{m\in\mathbb{Z}} \hat{f}(m) = \int_{\mathbb{R}} f(x)\,dx + \sum_{m\neq 0} \int_{\mathbb{R}} f(x)e(-mx)\,dx.$$

Proof. Define the periodized function $F(x) = \sum_{n\in\mathbb{Z}} f(x+n)$. Then the sum we're interested in is $F(0)$. It's easy to verify that $\hat{F}(n) = \hat{f}(n)$ for every $n \in \mathbb{Z}$. Hence, by Fourier inversion, $F(0) = \sum_{n\in\mathbb{Z}} \hat{F}(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n)$. □

Theorem 2 (Euler-Maclaurin formula)**.**

$$\sum_{n\in\mathbb{Z}} f(n) = \int_{\mathbb{R}} f(x)\,dx - \int_{\mathbb{R}} f'(x)s(x)\,dx$$

where $s(x) = 1/2 - x + \lfloor x \rfloor$. (This function $s(x)$ is sometimes called the "sawtooth" function.)

Proof. Using Riemann-Stieltjes integration and integration by parts we have

$$\begin{aligned}\sum_{n\in\mathbb{Z}} f(n) &= \int_{-\infty}^{\infty} f(x)\,d\lfloor x \rfloor \\ &= \int_{\mathbb{R}} f(x)\,dx + \int_{-\infty}^{\infty} f(x)\,ds(x) \\ &= \int_{\mathbb{R}} f(x)\,dx - \int_{\mathbb{R}} f'(u)s(x)\,dx.\end{aligned}$$

□

Remark 1. In the general Euler-Maclaurin formula one integrates by parts k times rather than just once. Doing this gives

$$\sum_{n\in\mathbb{Z}} f(n) = \int_{\mathbb{R}} f(x)\,dx + (-1)^k \int_{\mathbb{R}} f^{(k)}(u) s_k(x)\,dx$$

where s_k is a certain periodized degree k polynomial. These polynomials are called the Bernoulli polynomials.[1]

Remark 2. Usually the Euler-Maclaurin formula is stated (and applied) for sums over a bounded range, in which case there are boundary terms coming from the application of integration/summation by parts. The Poisson summation can also be applied for sums over bounded ranges by multiplying f by some cutoff function. This is also used in applications, although estimating the resulting oscillatory integrals can get messy because the Fourier transform of a non-continuous cutoff function will be rather badly behaved. To mitigate these difficulties, sometimes one may choose to use a smooth cutoff function instead.

3. Relationship between Theorem 1 and Theorem 2

With the formulas stated as they are above, it's easy to compare the two. Note that the term $\int_{\mathbb{R}} f(x)\,dx$ is present in both. Hence, both Poisson summation and Euler-Maclaurin can be thought of as giving an exact expression for the discrepancy between the sum $\sum_{n\in\mathbb{Z}} f(n)$ and the integral $\int_{\mathbb{R}} f(x)\,dx$. The Poisson summation formula says that this discrepancy is

$$\sum_{m\neq 0} \int_{\mathbb{R}} f(x) e(mx)\,dx$$

whereas the Euler-Maclaurin formula says it's

$$-\int_{\mathbb{R}} f'(x) s(x)\,dx.$$

To go from one expression to the other, we can use (1) integration by parts and (2) Fourier inversion. This is completely unsurprising since these are exactly what was used to prove the two formulas in the first place. The derivation is as follows.

$$\begin{aligned}\sum_{m\neq 0} \int_{\mathbb{R}} f(x) e(mx)\,dx &= -\sum_{m\neq 0} \int_{\mathbb{R}} f'(x) \frac{e(mx)}{2\pi i m}\,dx \\ &= -\lim_{M\to\infty} \int_{\mathbb{R}} f'(x) \sum_{0<|m|\le M} \frac{e(mx)}{2\pi i m}\,dx \\ &= -\int_{\mathbb{R}} f'(x) \lim_{M\to\infty} \sum_{0<|m|\le M} \frac{e(mx)}{2\pi i m}\,dx \\ &= -\int_{\mathbb{R}} f'(x) s(x)\,dx.\end{aligned}$$

In the last two equalities we have used the dominated convergence theorem and the fact that the Fourier series $\sum_{0<|m|\le M} \frac{e(mx)}{2\pi i m}$ converges boundedly to the sawtooth function $s(x)$ as $M \to \infty$ (except on a set of measure zero[2]).

[1] Actually the usual definition of the Bernoulli polynomials differs by a factor of $k!$.

[2] When x is an integer, this series sums to zero which is different from how we defined $s(x)$ earlier. Note that $s(x)$ is discontinuous at these points, and zero is value halfway between limiting value from the left and the limiting value from the right. This is a completely general phenomenon for the Fourier series of functions with jump discontinuities. The "correct way" to define $s(x)$ is to set it equal to zero at these points, so that it agrees with its Fourier series everywhere.

The derivation above gives another perspective on why the sawtooth function appears in the Euler-Maclaurin formula. If we integrate by parts k times, we can also derive expressions for the functions s_k discussed in Remark 1. Indeed, this gives

$$s_k(x) = \sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}} \frac{e(mx)}{(2\pi i m)^k}.$$

Note that for $k \geq 2$ the convergence is absolute and uniform, so the analysis is even easier to deal with in this scenario.

Lastly, we can consider the $k = 0$ case. Then we have the nonconvergent Fourier series expansion

$$s_0(x) = \sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}} e(mx).$$

In order to make sense of this rigorously, one needs to interpret s_0 as a generalized function (e.g. a tempered distribution). Writing down the Euler-Maclaurin formula in this case, we have

$$\sum_{n\in\mathbb{Z}} f(n) = \int_{\mathbb{R}} f(x)\,dx + \int_{\mathbb{R}} f(u)s_0(x)\,dx.$$

Here we think of f as a test function, so the formula states an equality for tempered distributions. The distribution on the left-hand side is the "Dirac comb", and the distribution on the right-hand side is Lebesgue measure plus s_0. Taking the Fourier transform of both sides leads to a simple interpretation of this. The Fourier transform of the Dirac comb is the Dirac comb (this is a restatement of the Poisson summation formula), the Fourier transform of Lebesgue measure is an atomic probability measure with a single atom at $x = 0$ (i.e. the Dirac delta function), and the Fourier transform of s_0 is simply the atomic measure with an atom at each nonzero integer point.

Department of Mathematics, University of Michigan, 530 Church St, Ann Arbor, MI 48109
Email address: adobner@umich.edu