WILF-EQUIVALENCE

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Given $\tau \in S_k$, a permutation $\sigma \in S_n$ *contains* τ (as a pattern) if there is a subsequence of σ whose elements are in the same relative order as τ. (Here we think of both τ and σ as ordered lists of numbers.) If this does not occur, then σ is said to *avoid* τ. For example, if $\tau = 213$ then σ avoids τ if there is no $1 \le a < b < c \le n$ such that $\sigma(b) < \sigma(a) < \sigma(c)$. In this context τ is referred to as a *pattern*. Pattern-avoidance has a natural equivalent definition in terms of permutation matrices: saying that σ avoids τ is equivalent to saying that the permutation matrix of σ does not contain the permutation matrix of τ as a submatrix.

Given a set of patterns Π, let

$$S_n(\Pi) \coloneqq \{\sigma \in S_n : \sigma \text{ avoids } \tau \text{ for all } \tau \in \Pi\}.$$

This is the set of Π-*avoiding patterns of length* n. If Π is a singleton set $\{\tau\}$ then we just write $S_n(\tau)$ to denote this set.

One of the goals in the study of pattern-avoiding permutations is to understand the size of $|S_n(\Pi)|$ for a given Π and n. For example, an early result along these lines (which is due to Knuth) is that for $\tau = 231$ we have $|S_n(\tau)| = C_n$, the nth Catalan number. In fact, $|S_n(\tau)| = C_n$ for *all* $\tau \in S_3$. This suggests the following question: for which sets of patterns Π_1 and Π_2 is it the case that $|S_n(\Pi_1)| = |S_n(\Pi_2)|$ for all $n \in \mathbb{N}$? If Π_1 and Π_2 have this property, then they are said to be *Wilf-equivalent*. This is written as $\Pi_1 \sim \Pi_2$. For singleton sets of patterns $\{\tau\}$ and $\{\tau'\}$ we write $\tau \sim \tau'$.

There are many known examples of Wilf-equivalence. The simplest of these are the trivial Wilf-equivalences that come from symmetry. These are best understood using permutation matrices: there is a natural action of D_4 (the symmetries of the square) on permutation matrices/patterns which preserves the notion of pattern avoidance. Consequently, for any pattern set Π, applying these symmetries gives up to 7 other pattern sets that are Wilf-equivalent to Π.

Other Wilf-equivalences are more difficult to come by. Restricting to the case of singleton sets, the known (nontrivial) Wilf-equivalences are as follows:

(1) (Stankova 1994 [8]) $1342 \sim 2413$.
(2) (Stankova & West 2002 [7]) $231 \oplus \tau \sim 312 \oplus \tau$ for any permutation τ. (Here $\oplus$ refers to the *direct sum* of permutations. The quickest definition is as follows: $\rho \oplus \pi$ is the permutation you get from taking the permutation matrices of ρ and π and putting them together into a block diagonal permutation matrix.)
(3) (Backelin, West, & Xin 2007 [1]) $12 \cdots k \oplus \tau$ and $k \cdots 21 \oplus \tau$ for any $k \in \mathbb{N}$ and any permutation τ.

The Wilf-equivalences (2) and (3) are actually examples of a stronger notion called *shape-Wilf-equivalence*. If Π_1, Π_2 are pattern sets let us write $\Pi_1 \sim_s \Pi_2$ if the sets are shape-Wilf-equivalent. (See [1] for a definition.) For singleton sets $\{\tau\}$ and $\{\tau'\}$ we write $\tau \sim_s \tau'$.

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Backelin et al. proved (see [1, Thm. 2.1]) a certain "prefix exchange" property for shape-Wilf-equivalence: if $\sigma \sim_s \sigma'$, then $\sigma \oplus \tau \sim_s \sigma' \oplus \tau$ for any τ. The Wilf-equivalences stated in (2) and (3) are consequences of this prefix exchange property and that $231 \sim_s 312$ and $12 \cdots k \sim_s k \cdots 1$ for all k.

1. Extending the results of Backelin et al.

There is a simple generalization of the (shape-)Wilf-equivalence in (3) which does not seem to have been explicitly stated in the literature. Let us introduce further notation. If Π and Γ are pattern sets, let $\Pi \oplus \Gamma := \{\pi \oplus \tau : \pi \in \Pi, \tau \in \Gamma\}$. The prefix exchange property of Backelin et al. can be extended to the following result using a straightforward modification of their proof.

Theorem 1 (see [2, Thm. 2.2])**.** *Let* Π_1, Π_2 *and* Γ *be sets of patterns. If* $\Pi_1 \sim_s \Pi_2$ *then* $\Pi_1 \oplus \Gamma \sim_s \Pi_2 \oplus \Gamma$*.*

This strengthening of the prefix exchange property is useful in light of the following result.

Theorem 2. *For any* $j, k \in \mathbb{N}$*, we have* $\{12 \cdots j, k \cdots 21\} \sim_s \{j \cdots 21, 12 \cdots k\}$*.*

Proof. The result $12 \cdots k \sim_s k \cdots 21$ of Backelin et al. was given a very transparent proof by Krattenthaler [5] (see also [9]). Krattenthaler's proof actually yields the stronger shape-Wilf-equivalence given in the theorem statement (although he did not explicitly state this). ☐

Hence, Theorem 1 and Theorem 2 together imply (for example) that $\{12 \cdots k \oplus \tau, j \cdots 21 \oplus \tau\} \sim_s \{j \cdots 21 \oplus \tau, 12 \cdots k \oplus \tau\}$. for any $j, k \in \mathbb{N}$ and any pattern τ. This is a generalization of the result of Backelin et al. stated in (3) above.

2. Computational testing for singleton pattern sets

Seeing as the only known examples of nontrivial Wilf-equivalence for singleton pattern sets are those given in (1), (2), and (3) above, it seems reasonable to make the following conjecture.

Conjecture 1. There are no further Wilf-equivalences for singleton pattern sets aside from those that are generated by the trivial equivalences and (1), (2), and (3).

This conjecture can be tested computationally. To do this, one can try to search for all Wilf-equivalences among patterns of length k for small values of k. For any given k, a straightforward algorithm for doing this is as follows.

(1) Generate the list of all $k!$ permutations in S_k. Apply the known Wilf-equivalences to reduce this to a list of candidate equivalence classes. The goal is then to check whether these candidate equivalence classes are all distinct.
(2) Select a representative from each candidate equivalence class. For each representative τ, compute $|S_n(\tau)|$ for $n = 1, 2, \ldots, N$ up to some threshold N.
(3) If the sequences computed in Step 2 are all distinct, then the classification is complete. If there are some duplicate sequences, then compute an additional term for each of the relevant sequences (i.e. increment N and repeat Step 2, but only do the computation for the duplicated sequences). Repeat this process until there are are no longer any duplicate sequences.

If this procedure terminates then this gives a complete classification of patterns of length k. If it does not terminate, then there must be new Wilf-equivalence which is not covered by any of the known cases.

In practice, carrying out this algorithm is quite computationally intensive. For $k \leq 7$, this computation was completed by Stankova and West [7, Fig. 9]. I tested the $k = 8$ case using some very efficient code written by William Kuszmaul [6] for calculating $|S_n(\tau)|$. This search did not yield any new Wilf-equivalences (i.e. Conjecture 1 remained valid).

The following table summarizes the results of the classification procedure for $k \leq 8$, listing the number of Wilf-equivalence classes for singleton patterns of length k.

Pattern length	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$
Wilf classes	1	1	1	3	16	91	595	4755

For $k = 9$ there are 42,681 candidate Wilf-equivalence classes. Completing the classification procedure for these is well within the realm of possibility for modern computers. For $k = 10$ one is probably better off looking for a different approach.

3. Structural perspective on Wilf-equivalence

3.1. Permutation classes (see [9]). The notion of Wilf-equivalence and pattern-avoidance can be treated in a more systematic way as follows. Let $S_{\text{fin}} := \bigcup_{n=0}^{\infty} S_n$. For any two permutations $\sigma, \pi \in S_{\text{fin}}$, let $\sigma \leq_c \pi$ if π contains σ as a pattern. It is easy to verify that this is a partial order on S. A *permutation class* is subset of S that is downward-closed with respect to this partial order.

The poset $(S_{\text{fin}}, \leq_c)$ satisfies the descending chain condition. This implies that upward-closed subsets $T \subseteq S$ are in bijection with antichains via the map $T \mapsto \min(T)$ that sends T to its minimal elements. Consequently, there is also a natural bijection between permutation classes and antichains (first take the complement of the permutation class; this is a bijection between downward-closed and upward-closed subsets of S_{fin}). Tracing these bijections, we see that any permutation class $\mathcal{C}$ has a uniquely associated antichain Π called a *basis* such that

$$\mathcal{C} = \{\sigma \in S_{\text{fin}} : \sigma \text{ avoids every pattern } \tau \in \Pi\}.$$

Extending our notation from earlier, let us define $S_{\text{fin}}(\Pi) = \bigsqcup_{n=0}^{\infty} S_n(\Pi)$. Also, for any permutation class $\mathcal{C}$, let $\mathcal{C}_n$ denote the subset of $\mathcal{C}$ that consists of the permutations in S_n.

With these new definitions in place, one can see that the notion of Wilf-equivalence is more naturally thought of as an equivalence relation on permutation classes. Given two permutation classes $\mathcal{C}$ and $\mathcal{D}$, they are *Wilf-equivalent* (denoted $\mathcal{C} \sim \mathcal{D}$) if there exists a bijection $\mathcal{C} \leftrightarrow \mathcal{D}$ that restricts to a bijection between $\mathcal{C}_n \leftrightarrow \mathcal{D}_n$ for all n.

3.2. Extending to arrays. Let us define an *array* to be a finitely supported doubly-infinite sequence $A = (a_{ij})_{i,j=1}^{\infty}$ where each a_{ij} is a nonnegative integer. Let Arr denote the set of all arrays. For any array A, let $\text{row}(A)$ be the finitely supported sequence given by $\text{row}(A)_i = \sum_{j=1}^{\infty} a_{ij}$ for all $i \geq 1$. That is, the ith term of $\text{row}(A)$ the sum of the entries in the ith row of A. Similarly, let $\text{col}(A)$ denote the column sums of A.

We can identify S_{fin} with a subset of Arr by sending any permutation to its associated permutation matrix and padding with zeros. This subset is precisely the set of arrays A such that $\text{row}(A) = \text{col}(A) = (1, 1, \ldots, 1, 0, 0, \ldots)$ for some number of 1's followed by infinitely many 0's.

The *interval minor order* on Arr is defined as follows.[1] Given arrays A, B we say $A \leq_{\text{im}} B$ if there exist sequences $1 = r_1 < r_2 < \dots$ and $1 = c_1 < c_2 < \dots$ such that

$$a_{ij} \leq \max_{\substack{i' \in [r_i, r_{i+1}) \\ j' \in [c_j, c_{j+1})}} b_{i'j'}.$$

In other words, $A \leq_{\text{im}} B$ means that it is possible to divide B into blocks, "contract" each block (by taking the maximum of the entries in that block), and obtain a matrix which is entrywise greater than or equal to A. This relation is a partial order on Arr.

Let us define an *array class* to be a downward closed set in Arr with respect to the interval minor order. We can define the notion of *interval minor avoidance* in the obvious way, and define $\text{Arr}(\Pi)$ to be the set of arrays that avoid every $A \in \Pi$ as an interval minor.

The interval minor order satisfies the descending chain condition, so the previous discussion for permutation classes also applies here. That is, each array class can be written as $\text{Arr}(\Pi)$ for a unique antichain Π in Arr.

Given array classes $\mathcal{C}$ and $\mathcal{D}$, let us say that $\mathcal{C}$ and $\mathcal{D}$ are *Wilf-equivalent* if there exists a bijection $\mathcal{C} \leftrightarrow \mathcal{D}$ that preserves all row sums and column sums. It would be interesting to know which Wilf-equivalences for permutation classes have analogues in the array class setting. As far as I know this has not been explored.

There are analogues of Theorem 2 in this setting which can be deduced from [5, Thm. 13]. Another example comes from my paper [3] (although it was not stated in this language).

Theorem 3 ([3, Thm. 9.1])**.** *For any $k \in \mathbb{N}$, $\text{Arr}((k-1)\cdots 21k)$ is Wilf-equivalent to $\text{Arr}(k \cdots 21)$.*

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[1] This order was defined by Fox [4] for 0-1 matrices.