GENERALIZED MONOID ALGEBRAS

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Abstract. In this note we discuss a general type of monoid algebra construction. This may be useful for formalizing various mathematical structures in computer proof assistants.

1. Motivation

Let R be a commutative ring and M a monoid. The *monoid algebra* $R[M]$ is defined by taking the set of all formal R-linear combinations of elements in M. The product of two elements $\sum_{\alpha \in M} c_\alpha \alpha$ and $\sum_{\beta \in M} d_\beta \beta$ is given by the "Cauchy product"

$$\left(\sum_{\alpha \in M} c_\alpha \alpha\right)\left(\sum_{\beta \in M} d_\beta \beta\right) = \sum_{\gamma \in M}\left(\sum_{\substack{\alpha,\beta \in M \\ \alpha \cdot \beta = \gamma}} c_\alpha d_\beta\right)\gamma.$$

This construction is quite general in that it produces a well-defined R-algebra[1] without requiring any further structure on M or R.

Example 1.1. A polynomial ring is a special case of a monoid algebra. For instance, $R[x]$ can be identified with the monoid algebra $R[M]$ where $M = \mathbb{N}_{\geq 0}$ is the monoid of nonnegative integers under addition. More generally, the multivariate polynomial ring $R[x_1, \ldots, x_n]$ can be identified with $R\left[\mathbb{N}_{\geq 0}^n\right]$.

There are other constructions of R-algebras that are similar to monoid algebras, but allow for more general formal sums.

Example 1.2. The formal power series ring $R[[x]]$ does not fit well into the monoid algebra framework because this ring allows for infinite sums of monomials.

The example above motivates a different general construction. Given a monoid M, let $R[[M]]$ denote the set of all (not necessarily finite) formal sums of elements in M with coefficients in R. This clearly has an R-module structure (it is equivalent to the set of all functions $M \to R$). However, multiplication is more delicate in this setting. In particular, the Cauchy product (equivalently, convolution of functions) may not be well-defined on $R[[M]]$.

Example 1.3. Consider the group $\mathbb{Z}$ under addition. We can think of $R[[\mathbb{Z}]]$ as the set of "generalized power series" $\sum_{n \in \mathbb{Z}} c_n x^n$. The product of two such series $\sum_{n \in \mathbb{Z}} c_n x^n$ and $\sum_{n \in \mathbb{Z}} d_n x^n$ will not necessarily be well-defined because the coefficient of x^0 in the product is given by the infinite sum $\sum_{n \in \mathbb{Z}} c_n d_{-n}$.

One way to handle such infinite formal sums is to introduce analysis or topology into the picture. For example, if $R = \mathbb{C}$ and M is a locally compact group, then $C_c(M)$ forms a well-defined algebra under convolution. Note that if M is a discrete group, then $C_c(M)$ reduces to $\mathbb{C}[M]$.

An alternative approach is to try to find purely algebraic conditions on M that guarantee that $R[[M]]$ is a well-defined algebra under the Cauchy product. We discuss this in the next section.

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[1] Here we use R-algebra to mean a unital associative algebra over R.

2. Finite decomposition monoids

As a starting point we can examine the example of $R[[x]]$ more closely. The key property that allows the product of power series to be well-defined is that for any fixed degree n, there are only finitely many pairs of monomials whose degrees sum to n. Equivalently, the monoid $\mathbb{N}_{\geq 0}$ has the property that any element can be decomposed as a sum in only finitely many ways. This motivates the following definition.

Definition 2.1. A monoid $(M, \cdot)$ is said to have the *finite decomposition property* if for every $m \in M$, the set $\{(a,b) \in M \times M \mid a \cdot b = m\}$ is finite. We call such a monoid a *finite decomposition monoid.*

The following proposition can be verified by a straightforward argument.

Proposition 2.2. *If R is a commutative ring and M is a finite decomposition monoid, then $R[[M]]$ is a well-defined R-algebra under the Cauchy product.*

The algebra $R[[M]]$ is sometimes called the *total (monoid) algebra* associated to M. This leads to a natural construction of multivariate power series rings.

Example 2.3. Fix an index set I. Let M be the monoid consisting of finitely supported elements of $\mathbb{N}^I_{\geq 0}$. Then M is a finite decomposition monoid. $R[[M]]$ is isomorphic to the ring of formal power series in the variables $\{x_i : i \in I\}$.

In some cases of interest, one may also want to study intermediate algebras between $R[M]$ and $R[[M]]$ where we allow for infinite formal sums but restrict the support of the sums in some way.

Example 2.4. Consider the subalgebra $R[[x_1, x_2, \ldots]]_{\mathrm{bdd}} \subset R[[x_1, x_2, \ldots]]$ consisting of all power series that have bounded total degree.[2] One may easily verify that the sum and product of two series with bounded degree are again series with bounded degree.

The above example is useful in, say, the theory of symmetric functions, where one studies the further subalgebra $\Lambda \subset R[[x_1, x_2, \ldots]]_{\mathrm{bdd}}$ consisting of those series that are invariant under permutations of the variables.

It is desirable to have a general algebraic framework for adapting the $R[[M]]$ construction to allow for restricted supports. In particular, even if the monoid M does *not* have the finite decomposition property, it may still be possible to construct well-defined algebras that are intermediate between $R[M]$ and $R[[M]]$.

Example 2.5 (Hahn series)**.** Let Γ be a totally ordered abelian group. Let $R[[\Gamma]]_{\text{well-ordered}}$ denote the set of formal sums $\sum_{\gamma \in \Gamma} c_\gamma \cdot \gamma$ such that the support $\{\gamma \in \Gamma : c_\gamma \neq 0\}$ is required to be a well-ordered subset of Γ. One can show that the product of two such series is well-defined and the product has well-ordered support. Hence $R[[\Gamma]]_{\text{well-ordered}}$ is an R-algebra, even though $R[[\Gamma]]$ may not be.

3. Finitely Decomposing Support Families

To guarantee that a subset of $R[[M]]$ forms a valid R-algebra, the collection of allowed supports must naturally obey certain closure properties. It is with this in mind that we introduce the following definition.

Definition 3.1. Let $(M, \cdot)$ be a monoid. A *support family* on M is a non-empty collection of subsets $\mathcal{F} \subseteq \mathcal{P}(M)$ satisfying the following four conditions:

1. **Contains identity:** $\{e\} \in \mathcal{F}$, where e is the identity element of M.
2. **Downward closed:** If $A \in \mathcal{F}$ and $B \subseteq A$, then $B \in \mathcal{F}$.
3. **Closed under finite unions:** If $A, B \in \mathcal{F}$, then $A \cup B \in \mathcal{F}$.
4. **Closed under monoid operation:** If $A, B \in \mathcal{F}$, then $A \cdot B = \{a \cdot b \mid a \in A, b \in B\}$ is in $\mathcal{F}$.

A support family $\mathcal{F}$ is said to be *finitely decomposing* if it also satisfies the following condition:

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[2] That is, $x_1 + x_2 + x_3 + \ldots$ is allowed, but $x_1 + x_1^2 + x_1^3 + \ldots$ is not.

(5) For any $A, B \in \mathcal{F}$ and any $m \in M$, the set of pairs $\{(a,b) \in A \times B \mid a \cdot b = m\}$ is finite.

The nonemptiness of $\mathcal{F}$ along with properties (2) and (3) is equivalent to saying that $\mathcal{F}$ is an ideal in the Boolean algebra $\mathcal{P}(M)$.

The definition of finitely decomposing support families is designed so as to make the following proposition hold.

Proposition 3.2. *Let R be a commutative ring, M a monoid, and $\mathcal{F}$ a finitely decomposing support family on M. Let $R[[M]]_{\mathcal{F}}$ denote the set of all formal sums $\sum_{\rho \in M} c_\rho \rho$ satisfying $\{\rho \in M : c_\rho \neq 0\} \in \mathcal{F}$. Then $R[[M]]_{\mathcal{F}}$ forms an R-algebra under the Cauchy product.*

Proof. The fact that $R[[M]]_{\mathcal{F}}$ is an R-module follows from (2) and (3). The fact that the product is well-defined and that $R[[M]]_{\mathcal{F}}$ is closed under multiplication follows from (4) and (5). Property (1) ensures the identity element is present. $\square$

This construction is nice in that it generalizes both the total monoid algebra $R[[M]]$ and ordinary monoid algebra $R[M]$.

Example 3.3. For any monoid M, it is easy to verify that the collection of all finite subsets of M forms a finitely decomposing support family. The resulting algebra $R[[M]]_{\text{fin}}$ is then just the monoid algebra $R[M]$.

It should be noted that if M is already a finite decomposition monoid, then *every* support family on M is finitely decomposing. This is useful because the notion of support families is well-behaved with respect to monoid homomorphisms. For example, we have the following definition and proposition.

Definition 3.4. Let $\phi : M \to N$ be a monoid homomorphism, and let $\mathcal{G} \subseteq \mathcal{P}(N)$. The *pullback* of $\mathcal{G}$ along ϕ is the collection of subsets $\phi^*\mathcal{G} = \{S \subseteq M \mid \phi[S] \in \mathcal{G}\}$.

Unfortunately, the "finitely decomposing" property of support families is not necessarily preserved under pullback,[3] but we do have the following result.

Proposition 3.5. *Let $\phi : M \to N$ be a monoid homomorphism, and let $\mathcal{G}$ be a support family on N. Then $\phi^*\mathcal{G}$ is a support family on M.*

This general proposition is sufficient to recover one of our earlier examples.

Example 3.6. Fix an index set I. Let M be the monoid consisting of finitely supported elements in $\mathbb{N}_{\geq 0}^I$. Let N be the monoid $\mathbb{N}_{\geq 0}$. Define $\phi : M \to N$ to be the "degree map". That is, $\phi\big((m_i)_{i \in I}\big) = \sum_{i \in I} m_i$. Let $\mathcal{G}$ be the collection of all finite subsets of $\mathbb{N}_{\geq 0}$. Then the pullback of $\mathcal{G}$ along ϕ consists of all subsets of M on which the degree map is bounded. Hence, we obtain the ring $R[[M]]_{\phi^*\mathcal{G}} \cong R[[x_i : i \in I]]_{\text{bdd}}$ discussed previously.

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[3] Consider a trivial homomorphism $\phi : \mathbb{Z} \to \{0\}$.